SH-III/Mathematics/304GE-3(T)/19

B.Sc. Semester III (Honours) Examination, 2018-19

MATHEMATICS

Course ID : 32114 **Course Code : SHMTH-304GE-3(T)**

Course Title : Algebra

Time: 2 Hours **Full Marks: 40**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

1. Answer *any five* questions: 2×5=10

(a) Find the values of $i^{3/4}$.

(b) Apply Descartes' rule of signs to find the nature of the roots of the equation
$x^4 + 7x^3 + 2x - 1 = 0$.

(c) If *a, b, c* are positive reals, not all equal, then prove that
$(a + b + c)(bc + ca + ab) > 9abc$.

(d) Let *S* be the set of all lines in 3-space. A relation *R* is defined on *S* by "*lRm* if and only if *l* lies on the plane of *m*" for $l, m \in S$. Examine whether *R* is an equivalence relation.

(e) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x^2 - 5$. Then find $f^{-1}(\{70\})$.

(f) Use Cayley-Hamilton theorem to find A^{-1} where $A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$.

(g) Find the dimension of the subspace S of $\mathbb{R}^4$,
where $S = \{(x, y, z, w) \in \mathbb{R}^4 : x + 2y - z + 2w = 0\}$.

(h) Examine whether the mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x + y, y + z, x + 1)$ is linear or not? Justify your answer.

2. Answer *any four* questions: 5×4=20

(a) Solve: $x^3 - 9x + 28 = 0$. 5

(b) (i) If the roots of the equation $x^3 + ax^2 + bx + c = 0$ be in G.P, show that $b^3 = a^3c$.

(ii) Show that $3x^5 - 4x^2 + 6 = 0$ has at least two imaginary roots. 3+2=5

(c) (i) If x, y, z are positive real numbers and $x + y + z = 1$, prove that
$8xyz \leq (1 - x)(1 - y)(1 - z) \leq \frac{8}{27}$.

(ii) If n be a positive integer >1, prove that $\left(\frac{n+1}{2}\right)^n > n!$ 3+2=5

(d) The matrix of $T: \mathbb{R}^3 \to \mathbb{R}^3$ relative to the ordered basis $\{(0,1,1),(1,0,1),(1,1,0)\}$ is $\begin{pmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{pmatrix}$. Find the linear transformation *T*. Find the matrix of *T* relative to the ordered basis $\{(2,1,1),(1,2,1),(1,1,2)\}$. 3+2=5

(e) (i) Use the theory of Congruences to prove that $17|(2^{3n+1} + 3 \cdot 5^{2n+1})$ for all positive integers *n*.

(ii) Let *a, b, c, m* be positive integers such that $ac \equiv bc \pmod{m}$ and $\gcd(c,m) = 1$ then prove that $a \equiv b \pmod{m}$. 3+2=5

(f) Solve if possible, the system of equations 5

$$x_1 + 2x_2 - x_3 = 0$$
$$-x_1 + x_2 + 2x_3 = 2$$
$$2x_1 + x_2 - 3x_3 = 2$$

3. Answer *any one* question: 10×1=10

(a) (i) If α, β are the roots of the equation $x^2 - 2x + 4 = 0$, prove that
$\alpha^n + \beta^n = 2^{n+1} cos\frac{n\pi}{3}$.

(ii) Show that the sum of the 99th power of the roots of the equation $x^5 = 1$ is zero.

(iii) Verify Cayley-Hamilton theorem for the matrix $\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Hence find A^{-1}.

3+3+4=10

(b) (i) Apply elementary row operations to reduce the following matrix to a row echelon matrix and find the rank of the matrix $\begin{pmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}$.

(ii) State first principle of mathematical induction. Apply this to prove that $3^{2n} = 8n - 1$ is divisible by 64. (4+1)+(2+3)=10

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